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The Translation Planes of Order 16 That Admit $PSL(2, 7)$

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1. INTRODUCTION

The main result of this article is:

(2.01) THEOREM. *A translation plane π of order 16 admits a collineation group isomorphic to $PSL(2, 7)$ if and only if π is derived from the semifield plane with kernel $GF(4)$ and admits affine elations.*

In [12] Lorimer constructs an extremely interesting translation plane of order 16 as follows: Consider the seven point projective plane $PG(2, 2)$ with points $\{1, 2, 3,..., 7\}$. The full collineation group of $PG(2, 2)$ is $PGL(3, 2)$ which is isomorphic to $PSL(2, 7)$. Moreover, using the fact that A_8 is isomorphic to $GL(4, 2)$, Lorimer corresponds, to every line $\{1, 2, 3\}$ of $PG(2, 2)$, two permutations (123) and (132) of A_7 and thus two 4×4 matrices over $GF(2)$. The seven lines of $PG(2, 2)$ give rise to 14 4×4 matrices M_i over $GF(2)$. If we take $y = 0$, $x = 0$, $y = x$, $y = xM_i$ as components we obtain a congruence and thus a translation plane π of order 16. Every collineation of $PG(2, 2)$ induces a collineation of the constructed plane π so that π admits $PSL(2, 7)$ as a collineation group. Lorimer actually uses the explicit isomorphism given in Dickson [1] (pp. 290–292) for the correspondence between A_8 and $GL(4, 2)$. This isomorphism gives rise to a translation plane of order 16 which is tangentially transitive with respect to a subplane of order 2.

Johnson and Ostrom [10] have noted that the plane of order 16 of Rahilly is also tangentially transitive and have shown the uniqueness of such planes. Thus the Lorimer and Rahilly planes are isomorphic.

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Walker [16] has also reconstructed the tangentially transitive plane of Lorimer–Rahilly and has, in fact, noted that this construction gives rise to another distinct translation plane π_1 of order 16 admitting $PSL(2, 7)$ as a collineation group. This plane π_1 may be constructed by using an isomorphism detween A_8 and $GL(4, 2)$ distinct from that given in Dickson.

Jha [6] has shown that any tangentially transitive plane is of order q^2 and may be derived from a semifield plane whose coordinate semifields possess middle nuclei containing $GF(q)$.

Thus, the Lorimer–Rahilly plane is derivable from the semifield planes of order 16 whose coordinate semifields are of dimension two over their middle nuclei. There is exactly one such semifield plane: the semifield plane of order 16 with kern $GF(4)$. Although it is not apparent from its construction in Walkers paper [16], the plane π_1 may also be derived from this same semifield plane.

Johnson [9] has completely determined the translation planes that may be obtained by the derivation of the unique semifield plane of order 16 and kernel $GF(4)$. It turns out that there are exactly three pairwise nonisomorphic planes; the unique tangentially transitive plane π_{tt} of Lorimer–Rahilly, a plane $\pi_{\mathscr{E}}$ (π_1 above) with the same orbit structure on l_∞ as π_{tt}, and a plane $\pi_{\mathscr{H}}$ which contains an invariant net of degree five. The planes π_{tt} and $\pi_{\mathscr{E}}$ both admit affine elations.

Our work then shows that if a translation plane π of order 16 admits a collineation group $\mathscr{G}$ such that $\mathscr{G} \simeq PSL(2, 7)$ then π is π_{tt} or $\pi_{\mathscr{E}}$ and conversely.

Aside from the exceptional properties of the Lorimer–Rahilly tangentially transitive plane π_{tj}, the planes π_{tt} and $\pi_{\mathscr{E}}$ are inteiesting in other ways. Johnson [8] has shown that if π is a translation plane of order q^2 that is derived from a semifield plane (by replacement of a net containing a shears axis) then if $q \neq 4$, π contains an *invariant* derivable net.

We see therefore that the planes π_{tt} and $\pi_{\mathscr{E}}$ show that this result cannot be extended to planes of order 16.

Actually, in some sense, the fact that a translation plane of order 16 can admit $PSL(2, 7)$ as a collineation group is perhaps the most remarkable single fact concerning these planes. That is, Hering and Ostrom [13], [14] have shown that in translation planes of order p^r the group E generated by elations is either

(1) elementary Abelian
(2) $SL(2, p^s)$
(3) $SL(2, 5), p = 3$
(4) $S_z(2^s)$, s odd

or

(5) $4 \nmid |E|$.

Moreover, Foulser [2] has shown that if $p \neq 2$, the group generated by Baer p-elements satisfies (1), (2), or (3).

Johnson and Ostrom [11] have also shown that if a nonsolvable group $\mathscr{G}$ contains only Baer involutions and the plane has dimension 2/kernel then the group generated by the Baer 2-elements in $\mathscr{G}$ is $SL(2, 2^s)$.

Thus, if a translation plane of order 16 admits $PSL(2, 7)$ then (1) its involutions are Baer (by Hering-Ostrom) and (2) the dimension/kernel is 4 (kernel is $GF(2)$) (by Johnson–Ostrom).

Therefore, the study and determination of the translation planes admitting $PSL(2, 7)$ forms the groundwork for studying the exceptional situations of Jha and Johnson mentioned above and the 4-dimensional analogue of the Johnson–Ostrom theory of Baer groups in translation planes of even order.

2. Translation Planes of Order 16 Admitting $PSL(2, 7)$

(2.1) Lemma. *Let π be a translation plane of order* 16. *The fixed point space of a collineation ρ of order* 7 *in the translation complement is always a subplane of order* 2.

Proof. Consider $\langle\rho\rangle \mid l_\infty$. If $\langle\rho\rangle \mid l_\infty = \langle 1\rangle$ then $\rho \in$ kernel π. But, $|$ kernel π $| = 15, 3$, or 1. The number of points on l_∞ is congruent to 3 mod 7. Thus, $\langle\rho\rangle$ must fix at least three points on l_∞ and fix at least three points on each of the fixed components. (This includes the origin and the point on l_∞.) Thus, ρ fixes a subplane π_1 pointwise. We cannot have $|\pi_1| = 4$ since otherwise 7 would divide $16 - 4 = 12$.

(2.2) Proposition. *Let π be a translation plane of order* 16. *Let $\mathscr{E}$* be an elementary abelian 2-group of order 4 which fixes some Baer subplane $\pi_{\mathscr{E}}$ pointwise. Let η be the net of π defined by $\pi_{\mathscr{E}}$. Let $\mathscr{H}$ be a collineation group of order 3 which fixes pointwise some Baer subplane of η. Then π is derivable and the associated derived plane $\bar{\pi}$ is either the unique semifield plane of order 16 with kernel $GF(4)$ or $\bar{\pi}$ is Desarguesian.

Proof. By Johnson ([7], Theorem (2.4)), π is derivable from a semifield plane $\bar{\pi}$ whose coordinate semifields either have their right nuclei $\supseteq GF(4)$. If $\bar{\pi}$ is not Desarguesian then the left nucleus (kernel in this case) is also isomorphic to $GF(4)$ (see e.g. Johnson [9] (1.1)).

(2.3) Proposition. *Let π be a translation plane of order* 16 *in which the translation complement contains a dihedral group S of order* 8 *which fixes a* 2*-space over $GF(2)$ pointwise. If the involutions of S are Baer then there is an elementary abelian subgroup of S of order* 4 *which fixes a Baer subplane pointwise.*

Proof. The 2-space fixed pointwise by S cannot be contained in a component $\mathscr{L}$ since if so S acts faithfully on $\mathscr{L}$ and the fixed points of S on $\mathscr{L}$ form a line of a Baer subplane which is fixed pointwise by some element of S. Thus, S is faithful and each element is fixed point free on the remaining 12 points of $\mathscr{L}$. But then 8 would have to divide 12.

Thus, S fixes a subplane π_0 of order 2 pointwise. Let $\langle\tau\rangle =$ center of S. Then S fixes the Baer subplane π fixed pointwise by τ. Since there are just two points of $\pi_\tau \cap l_\infty$ not in $\pi_0 \cap l_\infty$, there is a subgroup $\mathscr{E}$ of S of order 4 which fixes $\pi_\tau \cap l_\infty$ pointwise. Thus, $\mathscr{E}$ must fix π_τ pointwise. It follows by Foulser [3] that $\mathscr{E}$ must be elementary abelian.

(2.4) LEMMA. *If a translation plane of order* 16 *admits* $PSL(2, 7)$ *as a collineation group then the involutions are Baer and the orbits on* l_∞ *have lengths either*

(1) 14, 1, 1, 1

(2) 7, 7, 1, 1, 1

(3) 8, 7, 1, 1

or

(4) 8, 8, 1.

Proof. Since $PSL(2, 7)$ is simple, it is generated by its involutions, so by Hering [4], the involutions are Baer.

By (2.1), every element of order 7 has two orbits of length 7 on l_∞ and fixes three infinite points. By Galois (see e.g. Huppert [5] (p. 214)), $PSL(2, 7)$ must fix *pointwise* any set of cardinality $\leqslant 6$ which it fixes. Since $|PSL(2, 7)| = 2^3 \cdot 3 \cdot 7$, the possible orbit lengths on l_∞ therefore satisfy case (1), (2), (3), or (4).

(2.5) LEMMA. *Under the assumptions of* (2.4), *cases* (2) *or* (4) *cannot happen.*

Proof. Let $\mathscr{S}_2$ be a sylow 2-subgroup. In case (4), $\mathscr{S}_2$ must act regularly on the two orbits of length 8 (that is, the stabilizer of a point of one of these orbits has order 21). But, since the involutions are Baer, this is a contradiction.

In case (2), $\mathscr{S}_2$ must fix a point of each of the orbits of length 7 thereby fixing 5 points on the line at infinity. Since $\mathscr{S}_2$ fixes a subplane pointwise, $\mathscr{S}_2$ must fix a Baer subplane pointwise. But, $\mathscr{S}_2$ is dihedral of order 8 and by Foulser [3] a 2-group fixing a Baer subplane pointwise must be elementary abelian at characteristic two.

This proves (2.5).

Note that if $\mathscr{L}$ is a component fixed by $PSL(2, 7)$, we have case (1), (2), (3), or (4) on $\mathscr{L}$. However, case (4) cannot be the situation on $\mathscr{L}$ since there must be at least two fixed points ($\mathscr{O}$ and $\mathscr{L} \cap l_\infty$).

(2.6) LEMMA. *If we have case* (3) 8, 7, 1, 1 *on* l_∞ *we then have cases* (1) *or* (3) *on each of the fixed components.*

Proof. Let $x = 0, y = 0$ denote the fixed components. If we have case (2) on $y = 0$ or $x = 0$, then by the argument of (2.5), $\mathscr{I}_2$ must fix a Baer subplane pointwise and this is a contradiction as in the previous proof.

(2.7) LEMMA. *If we have case* (3) *on some fixed component* $\mathscr{L}$ *then the orbit of length* 7 *union the zero vector is a subspace of dimension* $3/GF(2)$.

Proof. If $\mathscr{G} \simeq PSL(2, 7)$ fixes $\mathscr{L}$ then $\mathscr{G} \simeq \mathscr{G}/\mathscr{L} \leqslant GL(4, 2)$. Every element of order 7 of $GL(4, 2)$ is conjugate and so fixes some 3-subspace (i.e., acts canonically as in $GL(3, 2)$). So, if we have orbits of length 7 and 8 on $\mathscr{L}$ and if the length 7 orbit (union $\mathcal{O}$) is not a 3-space then the orbit of length 8 contains a 7-orbit of an element of order 7 which is a subspace. Thus, either we have the proof to our assertion or the orbit of length $8 \cup \{\mathcal{O}\}$ is a set of nine vectors such that the deletion of any nonzero vector is a 3-subspace. But, this says that we have two vector subspaces V_1, V_2 of dimension $3/GF(2)$ that share seven vectors. Thus, $\dim(V_1 \cap V_2) \geqslant 3$ (i.e., $2^{\dim(V_1 \cap V_2)} \geqslant 7$) so that $V_1 = V_2$ which is a contradiction.

(2.8) LEMMA. *If we have case* (3) *on* l_∞, *if the restriction of* $\mathscr{G} = PSL(2, 7)$ *to* $x = 0$ *is of type* (1) *or* (3) *and if the restriction to* $y = 0$ *is of type* (1) *or* (3) *then* $\mathscr{G}$ *is represented by matrices of the form*

$$\left[\begin{array}{cccc|cccc}
1 & a_1 & a_2 & a_3 & & & & \\
b_1 & & & & & & & \\
b_2 & & A & & & & \mathcal{O} & \\
b_3 & & & & & & & \\
\hline
 & & & & 1 & c_1 & c_2 & c_3 \\
 & & & & d_1 & & & \\
 & & \mathcal{O} & & d_2 & & A & \\
 & & & & d_3 & & &
\end{array}\right]$$

where the various 3 *by* 3 *matrices* A *constitute a representation of* $\mathscr{G}$ *as* $\mathscr{G}L(3, 2)$ *and the other elements depend on* A *so that the* 4 *by* 4 *matrices in the upper left and lower right also give a representation of* $\mathscr{G}$.

Proof. Let λ be an element of order 7. Then λ fixes a component in the orbit of length 8. We can choose a basis so that $y = x$ is a component fixed by λ.

Consider the restriction of λ to one of the coordinate axes, say $y = 0$. In case (1), λ has an invariant 1-space which is also invariant under $\mathscr{G}$; in case (3), λ has an invariant 3-space which is invariant under $\mathscr{G}$. In fact $\langle\lambda\rangle$ is completely reducible and each coordinate axis (as a vector space) is the direct sum of an invariant 1-space and an invariant 3-space. (Since λ fixes three components and cannot fix pointwise a subplane of order greater than 4, a little counting establishes that the fixed point subspace on a fixed component must have dimension 1 and any other invariant subspace on the component must have dimension 3.)

If we represent $y = 0$ by ordered quadruples (x_1, x_2, x_3, x_4) we can choose a basis so that the invariant λ-space of dimension 1 is $\langle(1, 0, 0, 0)\rangle$ and the λ-space of simension 3 is the set of vectors for which $x_1 = 0$.

Thus the restriction of $\mathcal{G}$ to $y = 0$ (or $x = 0$) may be represented by matrices either of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & & & \\ b_2 & & A & \\ b_3 & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & & & \\ 0 & & A & \\ 0 & & & \end{pmatrix}$$

where the three by three matrices A are the matrices representing $\mathcal{G}$ as $\mathcal{G}L(3, 2)$. Thus the action of $\mathcal{G}$ on the plane can be represented by matrices of the form

$$\left[\begin{array}{cccc|cccc} 1 & a_1 & a_2 & a_3 & & & & \\ b_1 & & & & & & & \\ b_2 & & A & & & & & \\ b_3 & & & & & & & \\ \hline & & & & 1 & c_1 & c_2 & c_3 \\ & & & & d_1 & & & \\ & & & & d_2 & & A^\theta & \\ & & & & d_3 & & & \end{array}\right]$$

where θ is an automorphism of $\mathcal{G}$. The stabilizer of a point on l_∞ in one of the orbits of length 8 has order 21. The automorphism group of $PSL(2, 7)$ is isomorphic to $P\mathcal{G}L(2, 7)$; the automorphisms of $\mathcal{G}$ which fix each element of a group of order 21 correspond to the centralizer in $P\mathcal{G}L(2, 7)$ of a subgroup of order 21. This centralizer is trivial, i.e., $A^\theta = A$ for all A. This establishes Lemma (2.8).

(2.9) THEOREM. *If a translation plane of order* 16 *admits* $PSL(2, 7)$ *as a subgroup of the linear translation complement, then the orbits on* l_∞ *have lengths* 14, 1, 1, 1.

Proof. By Lemmas (2.5) and (2.6) we only need consider the possibility that the orbit decomposition on l_∞ might be 8, 7, 1, 1, 1. Hence we can use the representation of (2.8).

Now $\mathscr{G} \cong PSL(2, 7) \cong \mathscr{G}L(3, 2)$ is represented faithfully on the coordinate axes. Each three by three matrix over $\mathscr{G}F(2)$ is an element of $\mathscr{G}L(3, 2)$ and will appear as the submatrix A for one of the 8 by 8 matrices in (2.8). Let A be the following matrix of order 2:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The reader can verify that a necessary condition for

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 1 \\ b_3 & 0 & 0 & 1 \end{pmatrix}$$

to have order 2 is that $b_3 = 0$. (Look at the element in row 3, column 1 of the square of the above matrix.) Hence $\mathscr{G}$ contains an element in which

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

appears in the upper left hand corner and

$$\begin{pmatrix} 1 & c_1 & c_2 & c_3 \\ d_1 & 1 & 0 & 0 \\ d_2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

appears in the lower right hand corner for some a_i, b_i, c_i, d_i in $\mathscr{G}F(2)$.

The image of $y = x$ under this involution is a component $y = xM$ where M is the product of the above two matrices. It is immediate that M fixes $(0\ 0\ 0\ 1)$ and hence $y = xM$ has a nonzero vector in common with $y = x$. But $y = x$ and $y = xM$ must be distinct components since $y = x$ is in an orbit of length 8 and has a stabilizer of odd order. Distinct components can only share the zero vector. Thus we have a contradiction. We conclude that case (3) cannot occur on l_∞ so the theorem is proved.

(2.10) THEOREM. *A translation plane π of order* 16 *admits $PSL(2, 7)$ as a collination group if and only if π is derived from the semifield plane with kernel $GF(4)$ and admits elations.*

Proof. Let π admit $PSL(2, 7)$. By (2.11), we know the lengths of the orbits on l_∞ are 14, 1, 1, 1. We must also have cases (1), (2), (3) or (4) of (2.4) on the fixed components. By (2.3), a sylow 2-subgroup $\mathscr{S}_2$ contains an elementary abelian $\mathscr{E}$ subgroup of order 4 which fixes a Baer subplane π_τ pointwise. Let $\gamma \in$ normalizer of $\mathscr{E}$ and γ have order 3. Then by (2.11), γ must fix $\pi_\tau \cap l_\infty$ pointwise. By (2.4), we must have either case (1), (2) or (3) on fixed components so γ must fix some affine points and thus fixes a Baer subplane $\pi_{\mathscr{O}}$ pointwise. Since $\pi_{\mathscr{O}} \cap l_\infty = \pi_\tau \cap l_\infty$, π is derivable from the semifield plane of order 16 and kernel $GF(4)$ by (2.2).

By the structure of the orbits on l_∞, π must be one of the two planes that admit elations.

Since there does exist by Walker [16] at least two nonisomorphic planes of order 16 which admit $PSL(2, 7)$, we also have the converse.

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